

NOETHER-LASKER DECOMPOSITION OF COHERENT ANALYTIC SUBSHEAVES

BY
YUM-TONG SIU

In this paper we develop the theory of Noether-Lasker decomposition of coherent analytic subsheaves as an analogue of the algebraic Noether-Lasker decomposition of ideals in Noetherian rings. The decomposition can be described as follows: Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ in the sense of Grauert. For every point x of X , $\mathcal{S}_x$ as an $\mathcal{O}_x$ -submodule of $\mathcal{T}_x$ has a Noether-Lasker decomposition into primary $\mathcal{O}_x$ -submodules of $\mathcal{T}_x$. The radicals of these primary submodules are prime ideals of $\mathcal{O}_x$ which define subvariety-germs of X at x . These subvariety germs are pieced together to form global irreducible subvarieties of X which we call *associated subvarieties* of $\mathcal{S}$. A coherent subsheaf of $\mathcal{T}$ which has only one associated subvariety is called *primary*. We prove that every coherent analytic subsheaf can be represented as the intersection of "locally finite" primary subsheaves. This representation is what we call the Noether-Lasker decomposition of the coherent analytic subsheaf. If $(X, \mathcal{O})$ is Stein, then a coherent analytic proper subsheaf $\mathcal{S}$ of a coherent analytic sheaf $\mathcal{T}$ is primary if and only if $\Gamma(X, \mathcal{S})$ is a primary submodule of the $\Gamma(X, \mathcal{O})$ -module $\Gamma(X, \mathcal{T})$.

The Noether-Lasker decomposition of subsheaves is derived from the gap-sheaf theory of Thimm [4]. In part I of this paper we give an exposition of Thimm's theory of gap-sheaves by sheaf-theoretical methods. In part II of this paper we establish the Noether-Lasker decomposition of coherent analytic subsheaves.

Notations. All complex spaces in this paper are in the sense of Grauert [1, §1]. Suppose $(X, \mathcal{O})$ is a complex space. A *holomorphic function* on $(X, \mathcal{O})$ is an element of $\Gamma(X, \mathcal{O})$. A holomorphic function f *vanishes* at a point x of X if the germ of f at x is not a unit in $\mathcal{O}_x$. A *subvariety* in X is a set which locally is the set of points where a finite number of locally defined holomorphic functions vanish. The *ideal-sheaf* of a subvariety Y , denoted by $\text{Id } Y$, is the sheaf of germs of holomorphic functions vanishing at every point of Y . A complex space $(Z, \mathcal{H})$ is a *subspace* of $(X, \mathcal{O})$ if Z is a subvariety of X and there exists a coherent ideal-sheaf $\mathcal{I}$ on X such that $\mathcal{H} = (\mathcal{O}/\mathcal{I})|_Z$ and $\{z \mid z \in X, \mathcal{I}_z \neq \mathcal{O}_z\} = Z$. A *module-sheaf* on $(X, \mathcal{O})$ is an analytic subsheaf of $\mathcal{O}^p$ for some p . If $\mathcal{A}$ is an ideal-sheaf on $(X, \mathcal{O})$, then $\sqrt{\mathcal{A}}$ is the ideal-sheaf defined by $(\sqrt{\mathcal{A}})_x = \sqrt{\mathcal{A}_x}$, where $\sqrt{\mathcal{A}_x}$ is the radical of the ideal $\mathcal{A}_x$ in $\mathcal{O}_x$.

Received by the editors March 14, 1967 and, in revised form, August 24, 1967.

Suppose that $\mathcal{R}$ and $\mathcal{S}$ are analytic subsheaves of an analytic sheaf $\mathcal{T}$ and $\mathcal{A}$ is an ideal-sheaf on a complex space $(X, \mathcal{O})$. Denote by $\mathcal{R}:\mathcal{S}$ the ideal-sheaf defined as follows: for $x \in X$, $f \in (\mathcal{R}:\mathcal{S})_x$ if and only if $f \in \mathcal{O}_x$ and $f\mathcal{S}_x \subset \mathcal{R}_x$. Denote by $(\mathcal{S}:\mathcal{A})_{\mathcal{T}}$ or simply by $\mathcal{S}:\mathcal{A}$ the subsheaf of $\mathcal{T}$ defined as follows: for $x \in X$, $s \in (\mathcal{S}:\mathcal{A})_x$ if and only if $s \in \mathcal{T}_x$ and $\mathcal{A}_x s \subset \mathcal{S}_x$. If $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{A}$ are coherent, then $\mathcal{R}:\mathcal{S}$ and $\mathcal{S}:\mathcal{A}$ are coherent. If $s \in \Gamma(X, \mathcal{T})$ and $f \in \Gamma(X, \mathcal{O})$, then $\mathcal{R}:s$ denotes $\mathcal{R}:\mathcal{O}s$ and $(\mathcal{S}:f)_{\mathcal{T}}$ or simply $\mathcal{S}:f$ denotes $(\mathcal{S}:\mathcal{O}f)_{\mathcal{T}}$.

Suppose X and Y are two complex spaces, $\mathcal{T}$ is an analytic sheaf on X , and $\pi: X \rightarrow Y$ is a holomorphic map (i.e. a morphism of ringed spaces). Then $R^0\pi(\mathcal{T})$ denotes the zeroth direct image of $\mathcal{T}$ under π .

Suppose $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $r_1, \dots, r_n$ are positive numbers. Then $\Delta(x; r_1, \dots, r_n)$ denotes the polydisc $\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i - x_i| < r_i, 1 \leq i \leq n\}$. Suppose $\mathcal{F}$ is a sheaf on a topological space E , $x \in E$, U is an open neighborhood of x in E , and $s \in \Gamma(U, \mathcal{F})$. Then s_x denotes the germ of s at x and $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at x . If f is a (complex-valued) function on E , then f_x denotes the germ of f at x .

Suppose R is a Noetherian ring and E is an R -submodule of a finitely generated R -module F . Then $P(E, F)$ denotes the set of all associated nonunit prime ideals in the Noether-Lasker decomposition of E as a submodule of F .

I. Gap-sheaves.

DEFINITION 1. Suppose $\mathcal{S}$ is an analytic subsheaf of an analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and ρ is a nonnegative integer. The ρ th gap-sheaf of $\mathcal{S}$ in $\mathcal{T}$, denoted by $\mathcal{S}_{(\rho)}_{\mathcal{T}}$ or simply $\mathcal{S}_{(\rho)}$, is the analytic subsheaf of $\mathcal{T}$ defined as follows: for $x \in X$, $s \in (\mathcal{S}_{(\rho)})_x$ if and only if there exist an open neighborhood U of x in X , a subvariety A in U of dimension $\leq \rho$, and $t \in \Gamma(U, \mathcal{T})$ such that $t_x = s$ and $t_y \in \mathcal{S}_y$ for $y \in U - A$. Denote by $E(\mathcal{S}, \mathcal{T})$ the set $\{x \mid x \in X, \mathcal{S}_x \neq \mathcal{T}_x\}$ and $E^\rho(\mathcal{S}, \mathcal{T})$ denotes $E(\mathcal{S}, \mathcal{S}_{(\rho)})$.

DEFINITION 2. Suppose $\mathcal{S}$ is an analytic subsheaf of an analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and A is a subvariety of X . Then the gap-sheaf of $\mathcal{S}$ in $\mathcal{T}$ with respect to A , denoted by $\mathcal{S}[A]_{\mathcal{T}}$ or simply $\mathcal{S}[A]$, is the analytic subsheaf of $\mathcal{T}$ defined as follows: for $x \in X$, $s \in \mathcal{S}[A]_x$ if and only if there exist an open neighborhood U of x in X and $t \in \Gamma(U, \mathcal{T})$ such that $t_x = s$ and $t_y \in \mathcal{S}_y$ for $y \in U - A$.

THEOREM 1. Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and A is a subvariety of X . Then

$$\mathcal{S}[A] = \bigcup_{n=1}^{\infty} (\mathcal{S}:\mathcal{A}^n)_{\mathcal{T}},$$

where $\mathcal{A}$ is the ideal-sheaf of A , and hence is coherent.

Proof. Let $\mathcal{F} = \bigcup_{n=1}^{\infty} (\mathcal{S}:\mathcal{A}^n)_{\mathcal{T}} \subset \mathcal{T}$. $\mathcal{F}$ is coherent, because it is the union of an increasing sequence of coherent subsheaves of a coherent sheaf [1, Satz 8, §2].

Suppose $s \in \mathcal{S}[A]_x$ for some $x \in X$. Then there exist an open neighborhood U of

x in X and $t \in \Gamma(U, \mathcal{T})$ such that $t_x = s$ and $t_y \in \mathcal{S}_y$ for $y \in U - A$. Let $\mathcal{B} = (\mathcal{S}|U):t. E(\mathcal{B}, \mathcal{O}|U) \subset A \cap U$. By Hilbert Nullstellensatz [2, III.A.7] $\mathcal{A}_x^n \subset \mathcal{B}_x$ for some n . $s\mathcal{A}_x^n \subset s\mathcal{B}_x \subset \mathcal{S}_x$. $s \in \mathcal{F}$.

Suppose $s \in \mathcal{F}_x$. $s \in (\mathcal{S}:\mathcal{A}^n)_x$ for some n . There is an open neighborhood U of x and $t \in \Gamma(U, \mathcal{T})$ such that $t_x = s$ and $t(\mathcal{A}^n|U) \subset \mathcal{S}|U$. For $y \in U - A$, $\mathcal{A}_y^n = \mathcal{O}_y$. Hence $t_y \in \mathcal{S}_y$. $\mathcal{F} = \mathcal{S}[A]$. Q.E.D.

The following lemma is a particular case of [1, Hauptsatz I, §6] and it can be proved in a very elementary way.

LEMMA 1. Suppose X and Y are complex spaces, $\mathcal{F}$ is a coherent sheaf on X , and $\pi: X \rightarrow Y$ is a proper nowhere degenerate holomorphic map, then $R^0\pi(\mathcal{F})$ is coherent.

THEOREM 2. Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{H})$ and ρ is a nonnegative integer. Then $E^\rho(\mathcal{S}, \mathcal{T})$ is locally contained in a subvariety of dimension $\leq \rho$, i.e. for every $x \in X$ there exist an open neighborhood U of x in X and a subvariety A in U of dimension $\leq \rho$ such that $E^\rho(\mathcal{S}, \mathcal{T}) \cap U \subset A$.

Proof. Since the theorem is local in nature, we can suppose without loss of generality that $(X, \mathcal{H})$ is a subspace of an open subset G of $\mathbb{C}^n$. Let $\mathcal{O}$ be the structure sheaf of G and $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ be the trivial extensions of $\mathcal{S}$ and $\mathcal{T}$ on G respectively. We can further suppose without loss of generality that we have a sheaf-epimorphism $\lambda: \mathcal{O}^p \rightarrow \tilde{\mathcal{T}}$ on G . Let $\mathcal{M} = \lambda^{-1}(\tilde{\mathcal{S}})$. Then $E^\rho(\mathcal{S}, \mathcal{T}) = E^\rho(\mathcal{M}, \mathcal{O}^p)$. Hence we need only prove that

- (1) for every coherent module-sheaf $\mathcal{M} \subset \mathcal{O}^p$ on an open subset G of $\mathbb{C}^n$, $E^\rho(\mathcal{M}, \mathcal{O}^p)$ is locally contained in a subvariety of dimension $\leq \rho$.

We fix ρ and prove (1) by induction on n . For $n \leq \rho$, (1) is trivially true. Now suppose (1) is true when n is replaced by $n-1$. We are going to prove by induction on p that (1) is true when n is unreplaced.

(a) $p=1$. Because of the local nature of (1) we can suppose that G is connected. If $\mathcal{M}=0$, then (1) is trivial. So we can suppose $\mathcal{M} \neq 0$. $E(\mathcal{M}, \mathcal{O})$ is a proper subvariety of G . Take $x \in G$. We want to prove that $E^\rho(\mathcal{M}, \mathcal{O})$ is locally contained in a subvariety of dimension $\leq \rho$ at x . If $x \notin E(\mathcal{M}, \mathcal{O})$, then it is obviously true. So we suppose $x \in E(\mathcal{M}, \mathcal{O})$. There is a nonzero holomorphic function φ on some open neighborhood U of x such that φ vanishes on $E(\mathcal{M}, \mathcal{O}) \cap U$. Without loss of generality we can suppose that U is a polydisc $\Delta(x; r_1, \dots, r_n)$ and the projection $\pi: Y \rightarrow \Delta(\pi(x); r_1, \dots, r_{n-1})$ defined by $\pi(z_1, \dots, z_n) = (z_1, \dots, z_{n-1})$, where $Y = \{y \mid y \in U, \varphi(y) = 0\}$, makes Y an analytic cover over $\Delta(\pi(x); r_1, \dots, r_{n-1})$ [2, III.B.3].

By Hilbert Nullstellensatz, after shrinking U we can suppose without loss of generality that $\varphi^m \in \Gamma(U, \mathcal{M})$ for some m . $\mathcal{Q} = (\mathcal{M}/\mathcal{O}\varphi^m)|Y$ is a coherent analytic

sheaf on the complex space $(Y, \mathcal{K})$, where $\mathcal{K} = (\mathcal{O}/\mathcal{O}\varphi^m)|_Y$. $U \cap E^\rho(\mathcal{M}, \mathcal{O}) = E^\rho(\mathcal{Q}, \mathcal{K})$. Let $\mathcal{A} = R^0\pi(\mathcal{Q})$ and $\mathcal{B} = R^0\pi(\mathcal{K})$.

By Lemma 1, $\mathcal{A}$ is a coherent analytic subsheaf of the coherent analytic sheaf $\mathcal{B}$ on $\Delta(\pi(x); r_1, \dots, r_{n-1})$.

If $y \in E^\rho(\mathcal{Q}, \mathcal{K})$, then there exist a subvariety A of dimension $\leq \rho$ in an open neighborhood B of y in Y and $t \in \Gamma(B, \mathcal{K})$ such that $t_z \in \mathcal{Q}_z$ for $z \in B - A$ and $t_y \notin \mathcal{Q}_y$. Let $\pi^{-1}(\pi(y)) = \{y^0, \dots, y^l\}$, where $y^0 = y$, and B^i be disjoint open neighborhoods of y^i , $0 \leq i \leq l$, in Y , such that $B^0 \subset B$. Since φ is proper, there is an open neighborhood C of $\pi(y)$ in $\Delta(\pi(x); r_1, \dots, r_{n-1})$ such that $\pi^{-1}(C) \subset \bigcup_{i=0}^l B^i$. Define $t^* \in \Gamma(\pi^{-1}(C), \mathcal{K})$ as follows: $t^*|_{\pi^{-1}(C) \cap B^0} = t|_{\pi^{-1}(C) \cap B^0}$ and $t^*|_{\pi^{-1}(C) \cap B^i} = 0$ for $1 \leq i \leq l$. t^* induces $t' \in \Gamma(C, \mathcal{B})$. $t'_z \in \mathcal{A}_z$ for $z \in C - \pi(A)$ and $t'_{\pi(y)} \notin \mathcal{A}_{\pi(y)}$. Since $C \cap \pi(A)$ is a subvariety of dimension $\leq \rho$ in C , $\pi(y) \in E^\rho(\mathcal{A}, \mathcal{B})$. Since y is an arbitrary point in $E^\rho(\mathcal{Q}, \mathcal{K})$, $E^\rho(\mathcal{Q}, \mathcal{K}) \subset \pi^{-1}(E^\rho(\mathcal{A}, \mathcal{B}))$. Let $0 < s_i < r_i$, $1 \leq i \leq n-1$, and $\mathcal{R}$ be the structure sheaf of $D = \Delta(\pi(x); s_1, \dots, s_{n-1})$. Then there is a sheaf-epimorphism $\eta: \mathcal{R}^q \rightarrow \mathcal{B}|_D$.

$E^\rho(\mathcal{A}, \mathcal{B}) \cap D = E^\rho(\eta^{-1}(\mathcal{A}|_D), \mathcal{R}^q)$. By induction hypothesis $E^\rho(\eta^{-1}(\mathcal{A}|_D), \mathcal{R}^q)$ is locally contained in a subvariety of dimension $\leq \rho$. There exists a subvariety Z of dimension $\leq \rho$ in a polydisc $W = \Delta(\pi(x); t_1, \dots, t_{n-1}) \subset D$ such that $E^\rho(\mathcal{A}, \mathcal{B}) \cap W \subset Z$.

$$E^\rho(\mathcal{M}, \mathcal{O}) \cap \Delta(x; t_1, \dots, t_{n-1}, r_n)$$

$$= E^\rho(\mathcal{Q}, \mathcal{K}) \cap \Delta(x; t_1, \dots, t_{n-1}, r_n) \subset \pi^{-1}(E^\rho(\mathcal{A}, \mathcal{B}) \cap W) \subset \pi^{-1}(Z).$$

$\pi^{-1}(Z)$ is a subvariety of dimension $\leq \rho$ in $\Delta(x; t_1, \dots, t_{n-1}, r_n)$. The case $p=1$ is proved.

(b) The case of a general $p \geq 1$. $\mathcal{O}^p = \mathcal{O}^{p-1} \oplus \mathcal{O}$. Let $\alpha: \mathcal{O}^p \rightarrow \mathcal{O}^{p-1}$ be the projection onto the first summand and $\beta: \mathcal{O} \rightarrow \mathcal{O}^p$ be the injection from the second summand. Let $\mathcal{N} = \alpha(\mathcal{M})$ and $\mathcal{P} = \mathcal{M} \cap \beta(\mathcal{O})$. Take $x \in G$. Then by induction hypothesis and by (a) there exist subvarieties Z_1 and Z_2 of dimensions $\leq \rho$ in an open neighborhood U of x such that $U \cap E^\rho(\mathcal{N}, \mathcal{O}^{p-1}) \subset Z_1$ and $U \cap E^\rho(\mathcal{P}, \mathcal{O}) \subset Z_2$. It is readily checked that $U \cap E^\rho(\mathcal{M}, \mathcal{O}^p) \subset Z_1 \cup Z_2$. Q.E.D.

THEOREM 3. Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and ρ is a nonnegative integer. Then $\mathcal{S}_{[\rho]}$ is coherent and $E^\rho(\mathcal{S}, \mathcal{T})$ is a subvariety of dimension $\leq \rho$ in X .

Proof. First we prove the coherence of $\mathcal{S}_{[\rho]}$. Coherence is a local property. Take $x \in X$. By Theorem 2 there exists a subvariety A of dimension $\leq \rho$ in an open neighborhood U of x in X such that $U \cap E^\rho(\mathcal{S}, \mathcal{T}) \subset A$.

Since $\dim A \leq \rho$, $\mathcal{S}_{[\rho]}|_U = (\mathcal{S}|_U)[A]$. Hence $\mathcal{S}_{[\rho]}|_U$ is coherent by Theorem 1. $\mathcal{S}_{[\rho]}$ is coherent.

$E^\rho(\mathcal{S}, \mathcal{T}) = E((\mathcal{S}: \mathcal{S}_{[\rho]}), \mathcal{O})$ is a subvariety, because $\mathcal{S}: \mathcal{S}_{[\rho]}$ is coherent. Since $E^\rho(\mathcal{S}, \mathcal{T})$ is locally contained in a subvariety of dimension $\leq \rho$, $E^\rho(\mathcal{S}, \mathcal{T})$ is a subvariety of dimension $\leq \rho$ in X . Q.E.D.

COROLLARY. $\mathcal{S}_{[\rho]} = \mathcal{S}[E^\rho(\mathcal{S}, \mathcal{T})]$.

II. Noether-Lasker decomposition of subsheaves. Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space. Let $E^\rho(\mathcal{S}, \mathcal{T}) = \bigcup_{i \in I(\rho)} Y_i^\rho$ be the decomposition into irreducible branches. Then we call each nonempty Y_i^ρ , $\rho \geq 0$, $i \in I(\rho)$, an *associated subvariety* of $\mathcal{S}$ in $\mathcal{T}$ and denote the set of all associated subvarieties of $\mathcal{S}$ in $\mathcal{T}$ by $\mathcal{X}(\mathcal{S}, \mathcal{T})$. From the definition we see readily that $\mathcal{X}(\mathcal{S}, \mathcal{T})$ is locally finite. $\mathcal{S}$ is called a *primary subsheaf* of $\mathcal{T}$ if $\mathcal{S}$ has only one associated subvariety.

The following lemma is a well-known algebraic fact [5, Appendix, Chapter IV]:

LEMMA 2. Suppose R is a Noetherian ring and N is an R -submodule of a finitely generated R -module M . A prime ideal P in R is an associated prime ideal in the Noether-Lasker decomposition of N as a submodule of M if and only if $P = \sqrt{(N:f)}$ for some $f \in M$.

THEOREM 4. Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and $x \in X$. Let $\{X_i^\rho \mid \rho \geq 0, i \in J(\rho)\}$ be the set of all associated subvarieties of $\mathcal{S}$ passing through x , where $\dim X_i^\rho = \rho$, $i \in J(\rho)$, and suppose $(\text{Id } X_i^\rho)_x = \bigcap_{j \in K(\rho, i)} P_{ij}^\rho$ is the decomposition into prime ideals. Then $\{P_{ij}^\rho \mid \rho \geq 0, i \in J(\rho), j \in K(\rho, i)\} = P(\mathcal{S}_x, \mathcal{T}_x)$.

Proof. Suppose $P \in P(\mathcal{S}_x, \mathcal{T}_x)$ and $\dim P = \rho$. Then $P = \sqrt{(\mathcal{S}_x:f)}$ for some $f \in \mathcal{T}_x$ by Lemma 2. P defines a subvariety V of dimension ρ in an open neighborhood D of x in X . We can suppose after a shrinking of D that there exists $g \in \Gamma(D, \mathcal{T})$ such that $g_x = f$ and $\text{Id } V = \sqrt{((\mathcal{S}|D):g)}$. This implies that

$$\{y \mid y \in D, g_y \in \mathcal{S}_y\} = D - V.$$

Hence $V \subset E^\rho(\mathcal{S}, \mathcal{T})$. Since $\dim V = \rho$ and $\dim E^\rho(\mathcal{S}, \mathcal{T}) \leq \rho$, $P = P_{ij}^\rho$ for some $i \in J(\rho)$ and some $j \in K(\rho, i)$.

Fix $\rho \geq 0$ and $i \in J(\rho)$. By definition X_i^ρ is an irreducible branch of $E^\sigma(\mathcal{S}, \mathcal{T})$ for some $\sigma \geq \rho$. Let U be a Stein open neighborhood of x in X such that $U \cap X_i^\rho = \bigcup_{j \in K(\rho, i)} X_{ij}^\rho$ is the decomposition into irreducible branches and $P_{ij}^\rho = (\text{Id } X_{ij}^\rho)_x$, $j \in K(\rho, i)$. Fix $j \in K(\rho, i)$. Let Z^1 be the union of irreducible branches of $E^\sigma(\mathcal{S}, \mathcal{T}) \cap U$ other than X_{ij}^ρ and let

$$Z = Z^1 \cup (E^{\rho-1}(\mathcal{S}, \mathcal{T}) \cap U).$$

Take $y \in X_{ij}^\rho - Z$.

$$(\mathcal{S}|U)[X_{ij}^\rho]_y = \mathcal{S}_{[\sigma]y} \neq \mathcal{S}_y.$$

Since $(\mathcal{S}|U)[X_{ij}^\rho]$ is generated by global sections [1, Satz 4, §2], there exists $t \in \Gamma(U, (\mathcal{S}|U)[X_{ij}^\rho])$ such that $t_y \notin \mathcal{S}_y$.

Let $Y = E(\mathcal{S}|U, (\mathcal{S}|U) + (\mathcal{O}|U)t)$. Since X_{ij}^ρ is irreducible, if $Y \neq X_{ij}^\rho$, then $\dim Y < \rho$ and $t_y \in \mathcal{S}_{[\rho-1]y} = \mathcal{S}_y$ (contradiction). Hence $Y = X_{ij}^\rho$. $P_{ij}^\rho = (\text{Id } X_{ij}^\rho)_x = (\text{Id } Y)_x = \sqrt{(\mathcal{S}_x:t_x)}$. By Lemma 2, $P_{ij}^\rho \in P(\mathcal{S}_x, \mathcal{T}_x)$. Q.E.D.

This theorem gives us a characterization of associated subvarieties and tells us that the subvariety-germs defined by associated prime ideals in the Noether-Lasker decomposition of the stalks of $\mathcal{S}$ can be pieced together to form global subvarieties.

COROLLARY 1. *If $Y \in \mathcal{X}(\mathcal{S}, \mathcal{T})$, then $E(\mathcal{S}, \mathcal{S}[Y]) = Y$.*

Proof. Obviously $E(\mathcal{S}, \mathcal{S}[Y]) \subset Y$. Suppose $x \in Y$ and $P \in P((\text{Id } Y)_x, \mathcal{O}_x)$. Then by Theorem 4, $P \in P(\mathcal{S}_x, \mathcal{T}_x)$. By Lemma 2, $P = \sqrt{(\mathcal{S}_x : s)}$ for some $s \in \mathcal{T}_x$. $s \in \mathcal{S}[Y]_x - \mathcal{S}_x$. Q.E.D.

COROLLARY 2. *Suppose $\mathcal{R}$ is a coherent analytic subsheaf of $\mathcal{T}$ and $\mathcal{S} \subset \mathcal{R} \subset \mathcal{T}$ and $\mathcal{X}(\mathcal{S}, \mathcal{T}) = \{X^i \mid i \in I\}$. Then $\mathcal{X}(\mathcal{S}, \mathcal{R}) \subset \mathcal{X}(\mathcal{S}, \mathcal{T})$. Hence there is a subset J of I such that $E(\mathcal{S}, \mathcal{R}) = \bigcup \{X^i \mid i \in J\}$.*

Proof. Suppose $Y \in \mathcal{X}(\mathcal{S}, \mathcal{R})$. Take $y \in Y$ and $P \in P((\text{Id } Y)_y, \mathcal{O}_y)$. By Theorem 4, $P \in P(\mathcal{S}_y, \mathcal{R}_y)$. By Lemma 2, $P = \sqrt{(\mathcal{S}_y : s)}$ for some $s \in \mathcal{R}_y$. Since $s \in \mathcal{T}_y$, by Lemma 2, $P \in P(\mathcal{S}_y, \mathcal{T}_y)$.

By Theorem 4, $P \in P((\text{Id } X^i)_y, \mathcal{O}_y)$ for some $i \in I$ such that $y \in X^i$. Since the two irreducible subvarieties X^i and Y have a branch-germ in common at y , $X^i = Y$. Hence $\mathcal{X}(\mathcal{S}, \mathcal{R}) \subset \mathcal{X}(\mathcal{S}, \mathcal{T})$. The existence of J follows from

$$E(\mathcal{S}, \mathcal{R}) = \bigcup \{Y \mid Y \in \mathcal{X}(\mathcal{S}, \mathcal{R})\}. \quad \text{Q.E.D.}$$

THEOREM 5. *Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and A is a subvariety of X . Suppose $\mathcal{X}(\mathcal{S}, \mathcal{T}) = \{X^i \mid i \in I\}$, $x \in A$, and $I' = \{i \in I \mid x \in X^i\}$. Suppose*

$$P((\text{Id } X^i)_x, \mathcal{O}_x) = \{P_{ij} \mid j \in J_i\}, \quad i \in I'.$$

Let $\mathcal{S}_x = \bigcap \{Q_{ij} \mid i \in I', j \in J_i\}$ be a Noether-Lasker decomposition of $\mathcal{S}_x$, where the radical of Q_{ij} is P_{ij} , $i \in I', j \in J_i$, and let $K = \{i \in I', X^i \not\subset A\}$. Then

$$(\mathcal{S}[A])_x = \bigcap \{Q_{ij} \mid i \in K, j \in J_i\}.$$

Proof. Let $\mathcal{A} = \text{Id } A$. By Theorem 1

$$\begin{aligned} \mathcal{S}[A]_x &= \bigcap_{n=1}^{\infty} (\mathcal{S}_x : \mathcal{A}_x^n) = \bigcap_{n=1}^{\infty} \bigcap \{Q_{ij} : \mathcal{A}_x^n \mid i \in I', j \in J_i\} \\ &= \bigcap \left\{ \bigcap_{n=1}^{\infty} (Q_{ij} : \mathcal{A}_x^n) \mid i \in I', j \in J_i \right\}. \end{aligned}$$

For $i \in I' - K$, $P_{ij} \supset \mathcal{A}_x$ and hence $Q_{ij} : \mathcal{A}_x^n = \mathcal{T}_x$ for n sufficiently large. For $i \in K$, $P_{ij} \not\supset \mathcal{A}_x$ and hence $Q_{ij} : \mathcal{A}_x^n = Q_{ij}$ for every n . Therefore

$$\mathcal{S}[A]_x = \bigcap \{Q_{ij} \mid i \in K, j \in J_i\}. \quad \text{Q.E.D.}$$

COROLLARY. $\mathcal{X}(\mathcal{S}[A], \mathcal{T}) = \{X^i \mid X^i \not\subset A\}$ and

$$E(\mathcal{S}[A], \mathcal{T}) = \bigcup \{Y \mid Y \in \mathcal{X}(\mathcal{S}, \mathcal{T}), Y \not\subset A\}.$$

Proof. The first assertion follows from Theorems 4 and 5 and the second assertion follows from the first. Q.E.D.

LEMMA 3. Suppose $\mathcal{S} \subset \mathcal{R}$ are coherent analytic subsheaves of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$. Suppose $\mathcal{I}$ is the ideal-sheaf of $E(\mathcal{S}, \mathcal{R})$ and $x \in E(\mathcal{S}, \mathcal{R})$. Then there is a natural number k such that $((\mathcal{I}^k \mathcal{T} + \mathcal{S}) \cap \mathcal{R})_x = \mathcal{S}_x$.

Proof. $E(0, \mathcal{R}/\mathcal{S}) = E(\mathcal{S}, \mathcal{R})$. By Hilbert Nullstellensatz there is a natural number l such that $\mathcal{I}_x^l \subset (0: \mathcal{R}/\mathcal{S})_x$. By the Lemma of Artin-Rees [5, Theorem 4'; §2, Chapter VIII] there exists a natural number k such that $(\mathcal{I}^k(\mathcal{T}/\mathcal{S}) \cap (\mathcal{R}/\mathcal{S}))_x \subset (\mathcal{I}^l(\mathcal{R}/\mathcal{S}))_x$. Hence $((\mathcal{I}^k \mathcal{T} + \mathcal{S}) \cap \mathcal{R})_x = \mathcal{S}_x$. Q.E.D.

LEMMA 4. Suppose $\mathcal{S} \subset \mathcal{R}$ are coherent analytic subsheaves of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$. Then there exists a coherent analytic subsheaf $\mathcal{Q}$ of $\mathcal{T}$ such that $E(\mathcal{Q}, \mathcal{T}) = E(\mathcal{S}, \mathcal{R})$ and $\mathcal{Q} \cap \mathcal{R} = \mathcal{S}$.

Proof. Suppose $\mathcal{X}(\mathcal{S}, \mathcal{R}) = \{X^i \mid i \in I\}$. Let $\mathcal{I}_i$ be the ideal-sheaf of X^i . Take $x^i \in X^i$. By Lemma 3 and Corollary 1 to Theorem 4, there exists a natural number $k(i)$ such that

$$(2) \quad ((\mathcal{I}_i^{k(i)} \mathcal{T} + \mathcal{S}) \cap \mathcal{S}[X^i]_{\mathcal{R}})_{x^i} = \mathcal{S}_{x^i}, \quad i \in I.$$

Since $\mathcal{X}(\mathcal{S}, \mathcal{R})$ is locally finite, $\mathcal{Q} = \bigcap_{i \in I} (\mathcal{I}_i^{k(i)} \mathcal{T} + \mathcal{S})$ is a coherent analytic subsheaf of $\mathcal{T}$. Obviously $E(\mathcal{Q}, \mathcal{T}) \subset E(\mathcal{S}, \mathcal{R})$. We are going to prove that $\mathcal{Q} \cap \mathcal{R} = \mathcal{S}$.

From Corollaries 1 and 2 to Theorem 4 and (2) we conclude that

$$(3) \quad E(\mathcal{S}, (\mathcal{I}_i^{k(i)} \mathcal{T} + \mathcal{S}) \cap \mathcal{S}[X^i]_{\mathcal{R}}) \subset \bigcup \{X^j \mid j \in I, X^j \subsetneq X^i\}, \quad i \in I.$$

Obviously $\mathcal{S} \subset \mathcal{Q} \cap \mathcal{R}$. Let $Y = E(\mathcal{S}, \mathcal{Q} \cap \mathcal{R})$. By Corollary 2 to Theorem 4 there exists a subset J of I such that $Y = \bigcup \{X^i \mid i \in J\}$. Suppose $Y \neq \emptyset$. Then take $y \in Y$. Take a relatively compact open neighborhood U of y in X . Let $F = \{i \mid i \in J, X^i \cap U \neq \emptyset\}$. F is a finite set. Take $i \in F$ such that

$$\dim X^i = \max \{\dim X^j \mid j \in F\}.$$

Take an open neighborhood W of a point z of X^i in U such that

$$W \cap (\bigcup \{X^j \mid j \in I, X^j \not\supset X^i\}) = \emptyset.$$

$Y \cap W = X^i \cap W$. $(\mathcal{Q} \cap \mathcal{R})|_W \subset \mathcal{S}[X^i]_{\mathcal{R}}|_W$. By (3) $(\mathcal{I}_i^{k(i)} \mathcal{T} + \mathcal{S}) \cap \mathcal{S}[X^i]_{\mathcal{R}}|_W = \mathcal{S}|_W$. Hence

$$\mathcal{S}|_W \subset \mathcal{Q} \cap \mathcal{R}|_W \subset (\mathcal{I}_i^{k(i)} \mathcal{T} + \mathcal{S}) \cap \mathcal{S}[X^i]_{\mathcal{R}}|_W = \mathcal{S}|_W.$$

$z \notin Y$ (contradiction). Hence $Y = \emptyset$ and $\mathcal{S} = \mathcal{Q} \cap \mathcal{R}$.

Suppose $E(\mathcal{Q}, \mathcal{T}) \neq E(\mathcal{S}, \mathcal{R})$. Take $x \in E(\mathcal{S}, \mathcal{R}) - E(\mathcal{Q}, \mathcal{T})$. Then $\mathcal{Q}_x = \mathcal{T}_x$. $\mathcal{S}_x = \mathcal{Q}_x \cap \mathcal{R}_x = \mathcal{R}_x$, contradicting that $x \in E(\mathcal{S}, \mathcal{R})$. Hence $E(\mathcal{Q}, \mathcal{T}) = E(\mathcal{S}, \mathcal{R})$. Q.E.D.

LEMMA 5. Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and $\mathcal{X}(\mathcal{S}, \mathcal{T}) = \{X^i \mid i \in I\}$. Let $J = \{i \mid i \in I, X^i \text{ is maximal in } \mathcal{X}(\mathcal{S}, \mathcal{T})\}$. Then there exist a coherent analytic subsheaf $\mathcal{R}$ of $\mathcal{T}$ and primary subsheaves $\mathcal{Q}_i$ of $\mathcal{T}$, $i \in J$, such that (i) $E(\mathcal{R}, \mathcal{T}) = \bigcup_{i \in K} X^i$, where $K = I - J$, (ii) $E(\mathcal{Q}_i, \mathcal{T}) = X^i$, $i \in J$, and (iii) $(\bigcap_{i \in J} \mathcal{Q}_i) \cap \mathcal{R} = \mathcal{S}$.

Proof. For $i \in J$ let $Y^i = \bigcup \{X^j \mid j \in I \text{ and } j \neq i\}$ and define $\mathcal{Q}_i = \mathcal{S}[Y^i]$.

Then by Corollary to Theorem 5, $\{X^i\} = \mathcal{X}(\mathcal{Q}_i, \mathcal{T})$, $i \in J$. Hence $E(\mathcal{Q}_i, \mathcal{T}) = X^i$ and $\mathcal{Q}_i$ is a primary subsheaf of $\mathcal{T}$, $i \in J$. Take $y^i \in X^i - Y^i$, $i \in J$. Then $(\mathcal{Q}_i)_{y^i} = \mathcal{S}_{y^i}$, $i \in J$. Since $\mathcal{Q}_i \supset \mathcal{S}$, $i \in J$, we have

$$(4) \quad \left(\bigcap_{j \in J} \mathcal{Q}_j \right)_{y^i} = \mathcal{S}_{y^i}, \quad i \in J.$$

Since $\mathcal{X}(\mathcal{S}, \mathcal{T})$ is locally finite, $\bigcap_{i \in J} \mathcal{Q}_i$ is a coherent analytic subsheaf of $\mathcal{T}$. By (4) and Corollary 2 to Theorem 4 $E(\mathcal{S}, \bigcap_{i \in J} \mathcal{Q}_i) \subset \bigcup_{i \in K} X^i$.

Suppose $x \in \bigcup_{i \in K} X^i - E(\mathcal{S}, \bigcap_{i \in J} \mathcal{Q}_i)$. Then $x \in X^j$ for some $j \in K$ and $\mathcal{S}_x = \bigcap_{i \in J} (\mathcal{Q}_i)_x$. Let $L = \{i \mid i \in J, x \in X^i\}$. Since

$$P((\mathcal{Q}_i)_x, \mathcal{T}_x) = P((\text{Id } X^i)_x, \mathcal{O}_x), \quad i \in L,$$

$$P(\mathcal{S}_x, \mathcal{T}_x) \subset \bigcup_{i \in L} P((\text{Id } X^i)_x, \mathcal{O}_x).$$

Since

$$P((\text{Id } X^j)_x, \mathcal{O}_x) \cap \left(\bigcup_{i \in L} P((\text{Id } X^i)_x, \mathcal{O}_x) \right) = \emptyset,$$

Theorem 4, which asserts that $P((\text{Id } X^j)_x, \mathcal{O}_x) \subset P(\mathcal{S}_x, \mathcal{T}_x)$, is contradicted. Hence $E(\mathcal{S}, \bigcap_{i \in J} \mathcal{Q}_i) = \bigcup_{i \in K} X^i$. By Lemma 4 there exists a coherent analytic subsheaf $\mathcal{R}$ of $\mathcal{T}$ such that $(\bigcap_{i \in J} \mathcal{Q}_i) \cap \mathcal{R} = \mathcal{S}$ and $E(\mathcal{R}, \mathcal{T}) = \bigcup_{i \in K} X^i$. Q.E.D.

THEOREM 6 (NOETHER-LASKER DECOMPOSITION OF COHERENT SUBSHEAVES). Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$ and $\mathcal{X}(\mathcal{S}, \mathcal{T}) = \{X^i \mid i \in I\}$. Then for every $i \in I$, there exists a primary subsheaf $\mathcal{Q}_i$ of $\mathcal{T}$ such that $E(\mathcal{Q}_i, \mathcal{T}) = X^i$ and $\bigcap_{i \in I} \mathcal{Q}_i = \mathcal{S}$.

Proof. For $Y \in \mathcal{X}(\mathcal{S}, \mathcal{T})$ define the *depth* of Y in $\mathcal{X}(\mathcal{S}, \mathcal{T})$ to be $\sup \{l \mid \text{there exist } Y_j \in \mathcal{X}(\mathcal{S}, \mathcal{T}), 0 \leq j \leq l, \text{ such that } Y_0 = Y \text{ and } Y_j \subsetneq Y_{j+1} \text{ for } 0 \leq j < l\}$. If $Y_j \in \mathcal{X}(\mathcal{S}, \mathcal{T}), 0 \leq j \leq l$, and $Y_j \subsetneq Y_{j+1}, 0 \leq j < l$, then for $x \in Y_0$, $\dim_x Y_j < \dim_x Y_{j+1}$ (because Y_j is irreducible) and $l \leq \dim_x X$. So the depth of Y in $\mathcal{X}(\mathcal{S}, \mathcal{T})$ is finite for $Y \in \mathcal{X}(\mathcal{S}, \mathcal{T})$. For $i \in I$ denote the depth of X^i by d_i , and, for any nonnegative integer d , let $I_d = \{i \mid i \in I, d_i = d\}$, $J_d = \bigcup_{i \leq d} I_i$, and $K_d = I - J_d$.

We are going to prove by induction on d the following:

(5) For every $d \geq 0$ there exist primary subsheaves $\mathcal{Q}_i$ of $\mathcal{T}$ for $i \in I_d$ and a coherent analytic subsheaf $\mathcal{R}_d$ of $\mathcal{T}$ such that (i) $E(\mathcal{Q}_i, \mathcal{T}) = X^i, i \in I_d$, (ii) $E(\mathcal{R}_d, \mathcal{T}) = \bigcup_{i \in K_d} X^i$, and (iii) $(\bigcap_{i \in J_d} \mathcal{Q}_i) \cap \mathcal{R}_d = \mathcal{S}$.

$J_0 = \{X^i \mid X^i \text{ is maximal in } \mathcal{X}(\mathcal{S}, \mathcal{T})\}$. By Lemma 5, (5) is true for $d=0$. Suppose (5) is proved for $0 \leq d \leq e$. Since $E(\mathcal{R}_e, \mathcal{T}) = \bigcup_{i \in K_e} X^i$ and

$$E(\mathcal{R}_e, \mathcal{T}) = \bigcup \{Y \mid Y \in \mathcal{X}(\mathcal{R}_e, \mathcal{T})\},$$

$\{X^i \mid i \in I_{e+1}\} = \{Y \in \mathcal{X}(\mathcal{R}_e, \mathcal{T}) \mid Y \text{ is maximal in } \mathcal{X}(\mathcal{R}_e, \mathcal{T})\}$. By Lemma 5 there exist primary subsheaves $\mathcal{Q}_i$ of $\mathcal{T}$, $i \in I_{e+1}$, and a coherent analytic subsheaf $\mathcal{B}$ of $\mathcal{T}$ such that (i) $E(\mathcal{Q}_i, \mathcal{T}) = X^i$, $i \in I_{e+1}$, (ii) $E(\mathcal{B}, \mathcal{T})$ is thin in $\bigcup_{i \in I_{e+1}} X^i$, and (iii) $(\bigcap_{i \in I_{e+1}} \mathcal{Q}_i) \cap \mathcal{B} = \mathcal{R}_e$. Hence $(\bigcap_{i \in I_{e+1}} \mathcal{Q}_i) \cap \mathcal{B} = \mathcal{S}$. Let

$$Z = \{Y \mid Y \in \mathcal{X}(\mathcal{B}, \mathcal{T}), Y \not\subset \bigcup_{i \in K_{e+1}} X^i\}.$$

By Corollary to Theorem 5, $E(\mathcal{B}[Z]_{\mathcal{T}}, \mathcal{T}) \subset \bigcup_{i \in K_{e+1}} X^i$. Let

$$V = E(\mathcal{S}, (\bigcap_{i \in J_{e+1}} \mathcal{Q}_i) \cap \mathcal{B}[Z]_{\mathcal{T}}).$$

Then $V \subset Z$. Since Z is thin in $\bigcup_{i \in I_{e+1}} X^i$, by Corollary 2 to Theorem 4, $V \subset \bigcup_{i \in K_{e+1}} X^i$.

By Lemma 4 there exists a coherent analytic subsheaf $\mathcal{H}$ of $\mathcal{T}$ such that $E(\mathcal{H}, \mathcal{T}) = V$ and $(\bigcap_{i \in J_{e+1}} \mathcal{Q}_i) \cap \mathcal{B}[Z]_{\mathcal{T}} \cap \mathcal{H} = \mathcal{S}$. Let $\mathcal{R}_{e+1} = \mathcal{B}[Z]_{\mathcal{T}} \cap \mathcal{H}$. Then $(\bigcap_{i \in J_{e+1}} \mathcal{Q}_i) \cap \mathcal{R}_{e+1} = \mathcal{S}$ and $E(\mathcal{R}_{e+1}, \mathcal{T}) \subset \bigcup_{i \in K_{e+1}} X^i$. Suppose $E(\mathcal{R}_{e+1}, \mathcal{T}) \neq \bigcup_{i \in K_{e+1}} X^i$. Take $x \in \bigcup_{i \in K_{e+1}} X^i - E(\mathcal{R}_{e+1}, \mathcal{T})$. $x \in X^j$ for some $j \in K_{e+1}$ and $\mathcal{S}_x = \bigcap_{i \in J_{e+1}} (\mathcal{Q}_i)_x$. Let $L = \{i \mid i \in J_{e+1}, x \in X^i\}$. Since $P((\mathcal{Q}_i)_x, \mathcal{T}_x) = P((\text{Id } X^i)_x, \mathcal{O}_x)$, $i \in L$, $P(\mathcal{S}_x, \mathcal{T}_x) \subset \bigcup_{i \in L} P((\text{Id } X^i)_x, \mathcal{O}_x)$. However,

$$P((\text{Id } X^j)_x, \mathcal{O}_x) \cap \left(\bigcup_{i \in L} P((\text{Id } X^i)_x, \mathcal{O}_x) \right) = \emptyset,$$

contradicting Theorem 4, which asserts $P((\text{Id } X^j)_x, \mathcal{O}_x) \subset P(\mathcal{S}_x, \mathcal{T}_x)$. Hence $E(\mathcal{R}_{e+1}, \mathcal{T}) = \bigcup_{i \in K_{e+1}} X^i$. The induction process is complete and (5) is proved.

We claim that $\mathcal{S} = \bigcap_{i \in I} \mathcal{Q}_i$. Obviously $\mathcal{S} \subset \bigcap_{i \in I} \mathcal{Q}_i$. Take $x \in X$.

$$F = \{i \mid i \in I, x \in X^i\}$$

is a finite set. Take $d \geq \max \{d_i \mid i \in F\}$. Then $x \notin \bigcup_{i \in K_d} X^i$. $(\mathcal{R}_d)_x = \mathcal{T}_x$.

$$\mathcal{S}_x = \left(\bigcap_{i \in J_d} (\mathcal{Q}_i)_x \cap (\mathcal{R}_d)_x \right)_x = \bigcap_{i \in J_d} (\mathcal{Q}_i)_x \supset \bigcap_{i \in I} (\mathcal{Q}_i)_x. \quad \text{Q.E.D.}$$

REMARK. The decomposition $\mathcal{S} = \bigcap_{i \in I} \mathcal{Q}_i$ is irredundant, i.e. $\mathcal{S} \neq \bigcap_{i \in I - \{j\}} \mathcal{Q}_i$ for any $j \in I$; for otherwise by Theorem 4 we have

$$\mathcal{X}(\mathcal{S}, \mathcal{T}) \subset \bigcup_{i \in I - \{j\}} \mathcal{X}(\mathcal{Q}_i, \mathcal{T}) = \{X^i \mid i \in I - \{j\}\}.$$

In general, $\mathcal{Q}_i$, $i \in I$, is not uniquely determined. For example, when $(X, \mathcal{O})$ is $\mathbb{C}^2$ with coordinate-functions z_1 and z_2 , then $(\mathcal{O}_{z_1}) \cap (\mathcal{O}_{z_1^2} + \mathcal{O}_{z_2}) = (\mathcal{O}_{z_1}) \cap (\mathcal{O}_{z_1} + \mathcal{O}_{z_2})^2$ are two different irredundant Noether-Lasker decompositions.

However, corresponding to the uniqueness of isolated ideal components in the usual Noether-Lasker decomposition in rings, we have the following:

A subset L of $\mathcal{X}(\mathcal{S}, \mathcal{T})$ is called an *isolated system of associated subvarieties* if $Y_1 \subset Y_2$, $Y_i \in \mathcal{X}(\mathcal{S}, \mathcal{T})$, $i=1, 2$, and $Y_1 \in L$ imply $Y_2 \in L$. If L is an isolated system of associated subvarieties, then $\bigcap \{ \mathcal{Q}_i \mid X^i \in L \}$ is unique, because it is equal to $\mathcal{S}[\bigcup \{ X^i \mid X^i \notin L \}]$ by Corollary to Theorem 5.

THEOREM 7. *Suppose $\mathcal{S}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{T}$ on a complex space $(X, \mathcal{O})$. If $\mathcal{S}$ is primary, then $\Gamma(X, \mathcal{S})$ is a primary $\Gamma(X, \mathcal{O})$ -submodule of $\Gamma(X, \mathcal{T})$. The converse is true if $(X, \mathcal{O})$ is Stein and $\Gamma(X, \mathcal{S}) \neq \Gamma(X, \mathcal{T})$.*

Proof. (i) Suppose $\mathcal{S}$ is primary. Let $\{Y\} = \mathcal{X}(\mathcal{S}, \mathcal{T})$. We are going to prove that $\Gamma(X, \mathcal{S})$ is a primary $\Gamma(X, \mathcal{O})$ -submodule of $\Gamma(X, \mathcal{T})$ with $\Gamma(X, \text{Id } Y)$ as its radical.

Take $f \in \Gamma(X, \text{Id } Y)$. Fix $y \in Y$. Since $E(\mathcal{S}, \mathcal{T}, \mathcal{O}) = Y$, by Hilbert Nullstellensatz $f_y^k \mathcal{T}_y \subset \mathcal{S}_y$ for some natural number k . $\mathcal{X}(\mathcal{S}, f^k \mathcal{T} + \mathcal{S}) \subset \{Y\}$ by Corollary 2 to Theorem 4. $y \notin E(\mathcal{S}, f^k \mathcal{T} + \mathcal{S})$ implies $E(\mathcal{S}, f^k \mathcal{T} + \mathcal{S}) = \emptyset$. $f^k \Gamma(X, \mathcal{T}) \subset \Gamma(X, \mathcal{S})$.

Suppose $g \in \Gamma(X, \mathcal{O}) - \Gamma(X, \text{Id } Y)$ and $s \in \Gamma(X, \mathcal{T})$ such that $gs \in \Gamma(X, \mathcal{S})$. For some $y \in Y$ g does not vanish at y . Then $s_y \in \mathcal{S}_y$. $\mathcal{X}(\mathcal{S}, \mathcal{O}s + \mathcal{S}) \subset \{Y\}$ by Corollary 2 to Theorem 4. $y \notin E(\mathcal{S}, \mathcal{O}s + \mathcal{S})$ implies $E(\mathcal{S}, \mathcal{O}s + \mathcal{S}) = \emptyset$. $s \in \Gamma(X, \mathcal{T})$.

(ii) Suppose $\Gamma(X, \mathcal{S})$ is a primary $\Gamma(X, \mathcal{O})$ -submodule of $\Gamma(X, \mathcal{T})$. Suppose $\Gamma(X, \mathcal{S}) \neq \Gamma(X, \mathcal{T})$ and $(X, \mathcal{O})$ is Stein. Let $P \subset \Gamma(X, \mathcal{O})$ be the radical of $\Gamma(X, \mathcal{S})$. P defines a subvariety Y in X . Clearly $E(\mathcal{S}, \mathcal{S}) \subset Y$ and $E(\mathcal{S}, \mathcal{S}) \neq \emptyset$.

We claim $P = \Gamma(X, \text{Id } Y)$. Take $f \in \Gamma(X, \text{Id } Y)$. Fix $y \in Y$. By Hilbert Nullstellensatz $f_y^k \in (\sum_{i=1}^l \mathcal{O}g_i)_y$ for some natural number k and some $g_1, \dots, g_l \in P$. $y \notin E((\sum_{i=1}^l \mathcal{O}g_i):f, \mathcal{O})$. By Cartan Theorem A there exists $h \in \Gamma(X, (\sum_{i=1}^l \mathcal{O}g_i):f)$ such that h does not vanish at y . $hf \in \Gamma(X, \sum_{i=1}^l \mathcal{O}g_i)$ and $h \notin P$. Let $\varphi: \mathcal{O}^l \rightarrow \sum_{i=1}^l \mathcal{O}g_i$ be the sheaf-epimorphism defined by $\varphi(t_1, \dots, t_l) = \sum_{i=1}^l t_i(g_i)_x$ for $x \in X$ and $(t_1, \dots, t_l) \in \mathcal{O}_x^l$. Since $H^1(X, \text{Ker } \varphi) = 0$ by Cartan Theorem B, $hf = \sum_{i=1}^l c_i g_i$ for some $c_1, \dots, c_l \in \Gamma(X, \mathcal{O})$. $hf \in P$. $h \notin P$ implies $f \in P$. Hence $P = \Gamma(X, \text{Id } Y)$.

Y is irreducible, for otherwise $Y = Y_1 \cup Y_2$ for some subvarieties $Y_1 \neq Y$ and $Y_2 \neq Y$ and $P = \Gamma(X, \text{Id } Y) = \Gamma(X, \text{Id } Y_1) \cap \Gamma(X, \text{Id } Y_2)$ with $\Gamma(X, \text{Id } Y_i) \neq \Gamma(X, \text{Id } Y)$, $i = 1, 2$.

Suppose $Z \in \mathcal{X}(\mathcal{S}, \mathcal{T})$. $Z \subset Y$. Suppose $Y \neq Z$. Then take $y \in Y - Z$. By Corollary 1 to Theorem 4, $E(\mathcal{S}, \mathcal{S}[Z], \mathcal{O}) = Z$. By Cartan Theorem A there exists $f \in \Gamma(X, \mathcal{S}:\mathcal{S}[Z])$ such that f does not vanish at y and there exists $s \in \Gamma(X, \mathcal{S}[Z]) - \Gamma(X, \mathcal{S})$. Hence $f \notin P$, $s \notin \Gamma(X, \mathcal{S})$, and $fs \in \Gamma(X, \mathcal{S})$. Contradiction. $\mathcal{X}(\mathcal{S}, \mathcal{T}) = \{Y\}$. $\mathcal{S}$ is primary. Q.E.D.

REMARKS. (i) Theorem 7 justifies the term *primary subsheaf*.

(ii) Theorem ' of [3] follows from Theorems 6 and 7.

The author wishes to thank Professor Robert C. Gunning for his suggestions and encouragements.

REFERENCES

1. H. Grauert, *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Inst. Hautes Études Sci., No. 5, Paris, 1960.
2. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
3. O. Forster, *Primärzelegung in Steinschen Algebren*, Math. Ann. **154** (1964), 307–329.
4. W. Thimm, *Lückengarben von kohärenten analytischen Modulgarben*, Math. Ann. **148** (1962), 372–394.
5. O. Zariski and P. Samuel, *Commutative algebra*, Van Nostrand, Princeton, N. J., 1958.

PURDUE UNIVERSITY,
LAFAYETTE, INDIANA
UNIVERSITY OF NOTRE DAME,
NOTRE DAME, INDIANA